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Asymptotic Formulae of Multivariate Bernstein Approximation

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We expand multivariate Bernstein approximation asymptotically in this paper. Our results generalize Bernstein's asymptotic formulae (*C. R. Acad. Sci. URSS* (1932), 86–92) to the multivariate setting. © 1992 Academic Press, Inc

INTRODUCTION

Bernstein approximation, the approximation of functions by using Bernstein polynomials, is one of the classic research topics but is a very rich one. It dates back to 1912 and thousands of research papers related to this topic have been published since then (cf. [8, 3, 4]). Recently, the multivariate version of Bernstein polynomials called polynomials in B-form has attracted much attention and has become one of the main tools in computer aided geometric design and in multivariate spline approximation for the representation and computation of piecewise polynomial functions. The theoretical results and applications of multivariate Bernstein polynomials are well summarized in [2]. These motivate us to study the multivariate Bernstein approximation. In this paper, we are interested in obtaining the multivariate generalization of the asymptotic formulae of Bernstein approximation (see [9] and [1] or [5]).

Since the multivariate version of Bernstein approximation has attracted only sporadic attention as yet, there is not much research on the generalization available in the literature. However, the bivariate generalization was considered by Stancu. (See, e.g., [6, 7].) But he did not obtain the asymptotic constant and only bivariate Bernstein polynomials defined on a standard triangle were studied.

We are going to expand asymptotically the multivariate Bernstein approximation on an arbitrary s-simplex in R^s, $s \geqslant 1$. To be precise, let $\mathbf{v}^0, ..., \mathbf{v}^s \in R^s$ be $s+1$ distinct points such that the volume of the s-simplex

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$\langle \mathbf{v}^0, ..., \mathbf{v}^s \rangle$ of $\mathbf{v}^0, ..., \mathbf{v}^s$ is positive. Denote $T = \langle \mathbf{v}^0, ..., \mathbf{v}^s \rangle$. Then for each $\mathbf{x} \in T$, there exist unique $\lambda_0, ..., \lambda_s$ such that

$$\mathbf{x} = \sum_{i=0}^{s} \lambda_i \mathbf{v}^i$$

with $\sum_{i=0}^{s} \lambda_i = 1$ and $\lambda_i \geqslant 0$, $i = 0, ..., s$.

This $(s+1)$-tuple $\lambda = (\lambda_0, ..., \lambda_s)$ is called the barycentric coordinate of $\mathbf{x}$ with respect to T. It is well known that any polynomial $p_n(\mathbf{x})$ of total degree $\leqslant n$ can be expressed by using the basic functions

$$B_\alpha(\lambda) := \frac{|\alpha|!}{\alpha!} \lambda^\alpha, \qquad \alpha \in \mathbb{Z}_+^{s+1} \text{ with } |\alpha| = n$$

in the form

$$p_n(\mathbf{x}) = \sum_{\substack{|\alpha| = n \\ \alpha \in \mathbb{Z}_+^{s+1}}} c_\alpha B_\alpha(\lambda), \qquad \mathbf{x} \in T,$$

which is called $p_n(\mathbf{x})$ in B-form as in [2]. Here, as usual, for any $\alpha = (\alpha_0, ..., \alpha_s) \in \mathbb{Z}_+^{s+1}$, $|\alpha| = \alpha_0 + \cdots + \alpha_s$ and $\alpha! = \alpha_0! \cdots \alpha_s!$. Also, $\lambda^\alpha = \lambda_0^{\alpha_0} \cdots \lambda_s^{\alpha_s}$.

Let $\mathbf{x}_\alpha = (1/|\alpha|) \sum_{i=0}^{s} \alpha_i \mathbf{v}^i$ for any $\alpha \in \mathbb{Z}_+^{s+1}$. For any continuous function $f(\mathbf{x})$ defined on T it is well known that the multivariate Bernstein polynomials

$$B_n(f; \mathbf{x}) := \sum_{\substack{|\alpha| = n \\ \alpha \in \mathbb{Z}_+^{s+1}}} f(\mathbf{x}_\alpha) B_\alpha(\lambda)$$

converge to $f(\mathbf{x})$ uniformly on T as $n \to \infty$.

Our problem is to find the asymptotic expansion of $B_n(f; \mathbf{x}) - f(\mathbf{x})$. We state our main results in the next section and prove them in the section after.

1. Statement of Main Results

Define the polynomials $S_\gamma^n(\mathbf{x})$ of degree γ by

$$S_\gamma^n(\mathbf{x}) = n^{|\gamma|} \sum_{\substack{|\alpha| = n \\ \alpha \in \mathbb{Z}_+^{s+1}}} (\mathbf{x}_\alpha - \mathbf{x})^\gamma B_\alpha(\lambda)$$

for $\gamma \in \mathbb{Z}_+^s$. Then $S_\gamma^n(\mathbf{x})$ satisfies the following recurrence relation.

THEOREM 1. *For* $j \in \{1, ..., s\}$ *and* $\gamma \in \mathbb{Z}^s_+$,

$$S^n_{\gamma+e^j}(\mathbf{x}) = \sum_{i=0}^{s} \lambda_i \sum_{\substack{\beta<\gamma \\ \beta\neq\gamma}} (\mathbf{v}^i - \mathbf{v}^0)^{\gamma-\beta} \binom{\gamma}{\beta} [n(v^i_j - x_j)\, S^n_\beta(\mathbf{x}) - S^n_{\beta+e^j}(\mathbf{x})].$$

Here $\beta \leqslant \gamma$ iff $\beta_j \leqslant \gamma_j$, $j = 1, ..., s$ and $\mathbf{x} = (x_1, ..., x_s)$, $\mathbf{v}^i = (v^i_1, ..., v^i_s)$, $i = 0, ..., s$. Let us assume that $f(\mathbf{x})$ is sufficiently smooth. Then $f(\mathbf{x}_\alpha)$ has the Taylor expansion at $\mathbf{x}$

$$\begin{aligned} f(\mathbf{x}_\alpha) &= \sum_{k=0}^{\infty} \frac{1}{k!} D^k_{\mathbf{x}_\alpha - \mathbf{x}} f(\mathbf{x}) \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{\substack{|\gamma|=k \\ \gamma\in\mathbb{Z}^s_+}} \binom{k}{\gamma} D^\gamma f(\mathbf{x})(\mathbf{x}_\alpha - \mathbf{x})^\gamma \end{aligned}$$

for any $\alpha \in \mathbb{Z}^{s+1}_+$ with $|\alpha| = n$. Thus,

$$\begin{aligned} B_n(f; \mathbf{x}) - f(\mathbf{x}) &= \sum_{\substack{|\alpha|=n \\ \alpha\in\mathbb{Z}^{s+1}_+}} f(\mathbf{x}_\alpha)\, B_\alpha(\lambda) - f(\mathbf{x}) \\ &= \sum_{\substack{\gamma\in\mathbb{Z}^s_+ \\ \gamma\neq 0}} \frac{1}{\gamma!} D^\gamma f(\mathbf{x}) \frac{1}{n^{|\gamma|}} S^n_\gamma(x) \\ &= \sum_{\substack{\gamma\in\mathbb{Z}^s_+ \\ |\gamma|\geqslant 2}} \frac{1}{\gamma!} D^\gamma f(\mathbf{x}) \frac{1}{n^{|\gamma|}} S^n_\gamma(\mathbf{x}) \end{aligned}$$

since Bernstein polynomials preserve linear polynomials, i.e., $S^n_\gamma(\mathbf{x}) = 0$ for $|\gamma| \leqslant 1$.

In order to obtain the asymptotic expansion of $B_n(f; \mathbf{x}) - f(\mathbf{x})$, we need to further our study on $S^n_\gamma(\mathbf{x})$. We have the following lemmas.

LEMMA 1. *Fix* $i_1, ..., i_l \in \{0, 1, ..., s\}$. *Let* $\beta \in \mathbb{Z}^{s+1}_+$ *be the multi-integer such that* $\alpha_{i_1} \cdots \alpha_{i_l} = \alpha^\beta$ *for* $\alpha = (\alpha_0, \alpha_1, ..., \alpha_s)$. *Then there exist positive integers* $C^\beta_\gamma = \prod_{j=0}^{s} [0; \gamma_j]\, \phi_{\beta_j}$, $\gamma \leqslant \beta$ *such that*

$$\sum_{\substack{|\alpha|=n \\ \alpha\in\mathbb{Z}^{s+1}_+}} \alpha_{i_1} \cdots \alpha_{i_l} B_\alpha(\lambda) = \sum_{\substack{\gamma\leqslant\beta \\ \gamma_j\geqslant 1 \text{ if } \beta_j\neq 0}} C^\beta_\gamma n(n-1)\cdots(n-|\gamma|+1)\, \lambda^\gamma,$$

where $\phi_k(t) := t^k$ *and* $[0; j]_\phi := [0, 1, ..., j]\, \phi_k$ *denotes the* j*th order divided difference of* ϕ_k.

Furthermore, let P_l^k be a collection of l-partitions of $\{1, ..., k\}$. Here, an l-partition of $\{1, ..., k\}$ is a set of index subsets $\{(I_1, ..., I_l)\colon I_i \cap I_j = \emptyset, i, j = 1, ..., l$, and $\bigcup_{i=1}^l I_i = \{1, ..., k\}\}$. Denote by $|P_l^k|$ the cardinality of P_l^k. Then

LEMMA 2. *Let $\beta^k = k\mathbf{e}^1$ and $\gamma^l = l\mathbf{e}^1$ with $l \leqslant k$. We have*

$$|P_l^k| = C_{\gamma^l}^{\beta^k} = [0; l]\, \phi_k.$$

Hence, for any $\gamma \leqslant \beta$ with $\gamma_j \geqslant 1$ if $\beta_j \neq 0$,

$$C_\gamma^\beta = \left| \bigotimes_{i=0}^{s} P_{\gamma_i}^{\beta_i} \right|,$$

where $\otimes_{i=0}^s P_{\gamma_i}^{\beta_i}$ denotes the tensor product of partition collections $P_{\gamma_i}^{\beta_i}$.

In particular, $|P_k^k| = 1$, $|P_1^k| = 1$, and $|P_{k-1}^k| = \binom{k}{2}$ for $k \geqslant 1$.

With the lemmas above, we can show the following

THEOREM 2. *For any $\gamma \in \mathbb{Z}_+^s$ with $|\gamma| \geqslant 2$,*

$$\frac{1}{n^{|\gamma|}} S_\gamma^n(\mathbf{x}) = \sum_{k=1}^{|\gamma|} \sum_{\substack{\beta^1, ..., \beta^k \in \mathbb{Z}_+^s \\ \beta^1 + \cdots + \beta^k = \gamma}} \frac{n(n-1)\cdots(n-k+1)}{n^{|\gamma|}} \prod_{i=1}^{k} \left(\sum_{j=0}^{s} \lambda_j (\mathbf{v}^j - \mathbf{x})^{\beta^i} \right).$$

Since $\sum_{j=0}^s \lambda_j (\mathbf{v}^j - \mathbf{x})^{e^i} = 0$ for any standard unit vector $\mathbf{e}^i \in \mathbb{Z}_+^s$, we have the following

COROLLARY. *For any $\gamma \in \mathbb{Z}_+^s$ with $|\gamma| \geqslant 2$,*

$$\frac{1}{n^{|\gamma|}} S_\gamma^n(\mathbf{x}) = \sum_{k=1}^{|\gamma|-2} \sum_{\substack{\beta^1, ..., \beta^k \in \mathbb{Z}_+^s \\ \beta^1 + \cdots + \beta^k = \gamma \\ |\beta^i| \geqslant 2, i = 1, ..., k}} \frac{n(n-1)\cdots(n-k+1)}{n^{|\gamma|}} \prod_{i=1}^{k} \left(\sum_{j=0}^{s} \lambda_j (\mathbf{v}^j - \mathbf{x})^{\beta^i} \right).$$

In particular, for $|\gamma| = 2$ and 3, we have

$$\frac{1}{n^{|\gamma|}} S_\gamma^n(\mathbf{x}) = \frac{1}{n^{|\gamma|-1}} \sum_{j=0}^{s} \lambda_j (\mathbf{v}^j - \mathbf{x})^\gamma.$$

For $\gamma \in \mathbb{Z}_+^s$ with $|\gamma| = 4$, we write $\gamma = \gamma(1) + \gamma(2) + \gamma(3) + \gamma(4)$ with $|\gamma(i)| = 1$ for $i = 1, 2, 3, 4$ and have

$$\frac{1}{n^{|\gamma|}} S_\gamma^n(\mathbf{x}) = \frac{1}{n^{|\gamma|-1}} \sum_{j=0}^{s} \lambda_j(\mathbf{v}^j - \mathbf{x})^\gamma + \frac{n(n-1)}{n^{|\gamma|}}$$

$$\times \left[\sum_{i=0}^{s} \lambda_i(\mathbf{v}^i - \mathbf{x})^{\gamma(1)+\gamma(2)} \sum_{i=0}^{s} \lambda_i(\mathbf{v}^i - \mathbf{x})^{\gamma(3)+\gamma(4)} \right.$$

$$+ \sum_{i=0}^{s} \lambda_i(\mathbf{v}^i - \mathbf{x})^{\gamma(1)+\gamma(3)} \sum_{i=0}^{s} \lambda_i(\mathbf{v}^i - \mathbf{x})^{\gamma(2)+\gamma(4)}$$

$$\left. + \sum_{i=0}^{s} \lambda_i(\mathbf{v}^i - \mathbf{x})^{\gamma(1)+\gamma(4)} \sum_{i=0}^{s} \lambda_i(\mathbf{v}^i - \mathbf{x})^{\gamma(2)+\gamma(3)} \right]$$

Therefore, we finally reach our conclusion.

THEOREM 3. *For any* $f \in C^{2k}(T)$,

$$\lim_{n\to\infty} n^k \left[B_n(f; \mathbf{x}) - f(\mathbf{x}) - \sum_{\substack{\gamma \in \mathbb{Z}_+^s \\ |\gamma| \leq 2k-1}} \frac{1}{\gamma!} \frac{S_\gamma^n(\mathbf{x})}{n^{|\gamma|}} D^\gamma f(\mathbf{x}) \right]$$

$$= \sum_{\substack{\gamma \in \mathbb{Z}_+^s \\ |\gamma| = 2k}} \frac{1}{\gamma!} \sum_{\substack{\beta^1, ..., \beta^k \in \mathbb{Z}_+^s \\ \beta^1 + \cdots + \beta^k = \gamma \\ |\beta^i| = 2,\, i = 1, ..., k}} \prod_{i=1}^{k} \left(\sum_{j=0}^{s} \lambda_j(\mathbf{v}^j - \mathbf{x})^{\beta^i} \right) D^\gamma f(\mathbf{x}).$$

When $T = [0, 1]$ with $\mathbf{v}^0 = 0$ and $\mathbf{v}^1 = 1$ and hence $\lambda_0 = 1 - x$ and $\lambda_1 = x$, the asymptotic expansion above becomes the Bernstein expansion in [1] or [5] since

$$\sum_{\substack{\gamma \in \mathbb{Z}_+^s \\ |\gamma| = 2k}} \frac{1}{\gamma!} \sum_{\substack{\beta^1, ..., \beta^k \in \mathbb{Z}_+^s \\ \beta^1 + \cdots + \beta^k = \gamma \\ |\beta^i| = 2,\, i = 1, ..., k}} 1 = \frac{(2k-1)!!}{(2k)!}$$

and

$$\sum_{j=0}^{1} \lambda_j(\mathbf{v}^j - \mathbf{x})^{\beta^i} = x(1-x), \qquad \forall i = 1, ..., k.$$

For $k = 1$ and $k = 2$, we have the following

COROLLARY. *For any* $f \in C^2(T)$,

$$\lim_{n\to\infty} n[B_n(f; \mathbf{x}) - f(\mathbf{x})] = \sum_{\substack{\gamma \in \mathbb{Z}_+^s \\ |\gamma| = 2}} \frac{1}{\gamma!} \left(\sum_{j=0}^{s} \lambda_j(\mathbf{v}^j - \mathbf{x})^\gamma \right) D^\gamma f(\mathbf{x}).$$

For any $f \in C^4(T)$,

$$\lim_{n\to\infty} n^2 \left[B_n(f; \mathbf{x}) - f(\mathbf{x}) - \sum_{\substack{\gamma \in \mathbb{Z}^s_+ \\ |\gamma| = 2, 3}} \frac{1}{n^{|\gamma|-1}\gamma!} \right.$$

$$\left. \times \left(\sum_{j=0}^{s} \lambda_j (\mathbf{v}^j - \mathbf{x})^\gamma \right) D^\gamma f(\mathbf{x}) \right]$$

$$= \sum_{\substack{\gamma \in \mathbb{Z}^s_+ \\ |\gamma| = 4}} \frac{1}{\gamma!} \sum_{\substack{\beta^1, \beta^2 \in \mathbb{Z}^s_+ \\ \beta^1 + \beta^2 = \gamma \\ |\beta^1| = |\beta^2| = 2}} \left(\sum_{j=0}^{s} \lambda_j (\mathbf{v}^j - \mathbf{x})^{\beta^1} \right)$$

$$\times \left(\sum_{j=0}^{s} \lambda_j (\mathbf{v}^j - \mathbf{x})^{\beta^2} \right) D^\gamma f(\mathbf{x}).$$

2. Proofs of Our Results

Proof of Theorem 1. Define a generating function g of $S^n_\gamma(\mathbf{x})$, $\gamma \in \mathbb{Z}^s_+$ by

$$g(\mathbf{t}; \mathbf{x}) = \sum_{\gamma \in \mathbb{Z}^s_+} \frac{S^n_\gamma(\mathbf{x})}{\gamma!} \left(\frac{\mathbf{t}}{n} \right)^\gamma, \qquad \mathbf{x}, \mathbf{t} \in \mathbb{R}^s.$$

Then

$$g(\mathbf{t}; \mathbf{x}) = \sum_{\substack{|\alpha| = n \\ \alpha \in \mathbb{Z}^{s+1}_+}} \sum_{\gamma \in \mathbb{Z}^s_+} \frac{(\mathbf{x}_\alpha - \mathbf{x})^\gamma \, \mathbf{t}^\gamma}{\gamma!} B_\alpha(\lambda)$$

$$= \sum_{\substack{|\alpha| = n \\ \alpha \in \mathbb{Z}^{s+1}_+}} e^{(\mathbf{x}_\alpha - \mathbf{x}) \cdot \mathbf{t}} B_\alpha(\lambda)$$

$$= e^{-\mathbf{x} \cdot \mathbf{t}} \sum_{\substack{|\alpha| = n \\ \alpha \in \mathbb{Z}^{s+1}_+}} \binom{n}{\alpha} e^{(1/n) \sum_{i=0}^{s} \alpha_i \mathbf{v}^i \cdot \mathbf{t}} \lambda^\alpha$$

$$= e^{-\mathbf{x} \cdot \mathbf{t}} \left(\sum_{i=0}^{s} \lambda_i e^{\mathbf{v}^i \cdot \mathbf{t}/n} \right)^n.$$

It follows that for each $j \in \{1, 2, ..., s\}$,

$$\frac{\partial g}{\partial t_j} = -x_j g + \frac{ng}{\sum_{i=0}^{s} \lambda_i e^{\mathbf{v}^i \cdot \mathbf{t}/n}} \sum_{i=0}^{s} \lambda_i \frac{v^i_j}{n} e^{\mathbf{v}^i \cdot \mathbf{t}/n}.$$

We obtain the following after some simplification:

$$e^{\mathbf{v}^0\cdot\mathbf{t}/n}\frac{\partial g}{\partial t_j}=\sum_{i=1}^{s}\lambda_i(e^{\mathbf{v}^0\cdot\mathbf{t}/n}-e^{\mathbf{v}^i\cdot\mathbf{t}/n})\frac{\partial g}{\partial t_j}$$
$$+x_j g\sum_{i=1}^{s}\lambda_i(e^{\mathbf{v}^0\cdot\mathbf{t}/n}-e^{\mathbf{v}^i\cdot\mathbf{t}/n})$$
$$-\sum_{i=0}^{s}\lambda_i v_j^i(e^{\mathbf{v}^0\cdot\mathbf{t}/n}-e^{\mathbf{v}^i\cdot\mathbf{t}/n})\,g.$$

Thus,

$$\frac{\partial g}{\partial t_j}=\sum_{i=1}^{s}\lambda_i(1-e^{(\mathbf{v}^i-\mathbf{v}^0)\cdot\mathbf{t}/n})\left[(x_j-v_j^i)\,g+\frac{\partial g}{\partial t_j}\right].$$

On the other hand,

$$\frac{\partial g}{\partial t_j}=\sum_{\substack{\gamma\in\mathbb{Z}_+^s\\ \gamma\geqslant e^j}}\frac{S_\gamma^n(\mathbf{x})}{(\gamma-\mathbf{e}^j)!}\left(\frac{\mathbf{t}}{n}\right)^{\gamma-\mathbf{e}^j}\frac{1}{n}$$
$$=\frac{1}{n}\sum_{\gamma\in\mathbb{Z}_+^s}\frac{S_{\gamma+\mathbf{e}^j}^n(\mathbf{x})}{\gamma!}\left(\frac{\mathbf{t}}{n}\right)^{\gamma}.$$

Since

$$1-e^{(\mathbf{v}^i-\mathbf{v}^0)\cdot\mathbf{t}/n}=-\sum_{\substack{\alpha\in\mathbb{Z}_+^s\\ \alpha\neq 0}}\frac{(\mathbf{v}^i-\mathbf{v}^0)^\alpha}{\alpha!}\left(\frac{\mathbf{t}}{n}\right)^{\alpha},$$

$$\sum_{i=1}^{s}\lambda_i(1-e^{(\mathbf{v}^i-\mathbf{v}^0)\cdot\mathbf{t}/n})\left[(x_j-v_j^i)\,g+\frac{\partial g}{\partial t_j}\right]$$
$$=\sum_{i=1}^{s}\lambda_i\sum_{\substack{\alpha,\beta\in\mathbb{Z}_+^s\\ \alpha\neq 0}}\left[(v_j^i-x_j)\,S_\beta^n(\mathbf{x})-\frac{S_{\beta+\mathbf{e}^j}^n(\mathbf{x})}{n}\right]\frac{(\mathbf{v}^i-\mathbf{v}^0)^\alpha}{\alpha!\,\beta!}\left(\frac{\mathbf{t}}{n}\right)^{\alpha+\beta}$$
$$=\sum_{i=1}^{s}\lambda_i\sum_{\gamma\in\mathbb{Z}_+^s}\sum_{\substack{\beta\leqslant\gamma\\ \beta\neq\gamma}}\left[(v_j^i-x_j)\,S_\beta^n(\mathbf{x})-\frac{S_{\beta+\mathbf{e}^j}^n(\mathbf{x})}{n}\right]$$
$$\times\binom{\gamma}{\beta}(\mathbf{v}^i-\mathbf{v}^0)^{\gamma-\beta}\frac{(\mathbf{t}/n)^\gamma}{\gamma!}.$$

Comparing the above expression with that of $\partial g/\partial t_j$, we immediately obtain the recurrence relation in Theorem 1. Hence, we have established Theorem 1.

When $T=\{(x_1, x_2)\colon 0\leqslant x_1,\ x_2\leqslant 1 \text{ and } 0\leqslant x_1+x_2\leqslant 1\}$, the recurrence relation is the same as the one obtained by Stancu in [6].

Proof of Lemma 1. By using the Newton interpolation formula,

$$\phi_k(t)=\sum_{i=1}^{k} [0;i]\,\phi_k t(t-1)\cdots(t-i+1).$$

Thus,

$$\begin{aligned}
\alpha_{i_1}\cdots\alpha_{i_l}=\alpha^\beta &= \prod_{i=0}^{s}\alpha_i^{\beta_i}\\
&=\prod_{i=0}^{s}\left(\sum_{j=1}^{\beta_i}[0;j]\,\phi_{\beta_i}\alpha_i(\alpha_i-1)\cdots(\alpha_i-j+1)\right)\\
&=\sum_{\substack{\gamma\leqslant\beta\\ \gamma_j\geqslant 1 \text{ if } \beta_j\neq 0}}\ \prod_{j=0}^{s}[0;\gamma_j]\,\phi_{\beta_j}\alpha_j(\alpha_j-1)\cdots(\alpha_j-\gamma_j+1).
\end{aligned}$$

Hence,

$$\begin{aligned}
\sum_{\substack{|\alpha|=n\\ \alpha\in\mathbb{Z}_+^{s+1}}}\alpha^\beta B_\alpha(\lambda) &= \sum_{\substack{\gamma\leqslant\beta\\ \gamma_j\geqslant 1 \text{ if } \beta_j\neq 0}}\ \prod_{j=0}^{s}[0;\gamma_j]\,\phi_{\beta_j}\\
&\quad\times\sum_{\substack{|\alpha|=n\\ \alpha\in\mathbb{Z}_+^{s+1}}}\ \prod_{j=0}^{s}\alpha_j(\alpha_j-1)\cdots(\alpha_j-\gamma_j+1)\,B_\alpha(\lambda)\\
&=\sum_{\substack{\gamma\leqslant\beta\\ \gamma_j\geqslant 1 \text{ if } \beta_j\neq 0}}\ \prod_{j=0}^{s}[0;\gamma_j]\,\phi_{\beta_j}n(n-1)\cdots(n-|\gamma|+1)\,\lambda^\gamma.
\end{aligned}$$

We now need to show that $\prod_{j=0}^{s}[0;\gamma_j]\,\phi_{\beta_j}$ is a positive integer. This can be done by using an induction argument. We leave the details to the interested reader. Actually, this fact follows from Lemma 2. Thus, we have completed the proof of Lemma 1.

Proof of Lemma 2. We use induction to prove this lemma. It is clear that $|P_1^1|=1$ and $|P_1^{k+1}|=1$. We may assume that $|P_i^j|=[0;i]\,\phi_j$ for $1\leqslant i\leqslant j\leqslant k$. For $|P_l^{k+1}|$, we consider $[0;l]\,\phi_{k+1}$. By a property of divided differences and induction hypothesis,

$$
\begin{aligned}
[0; l]\,\phi_{k+1} &= \sum_{j=0}^{l} [0; j]\,\phi_k [j; k]\,\phi_1 \\
&= [0; l-1]\,\phi_k + [0; l]\,\phi_k l \\
&= |P_{l-1}^k| + l|P_l^k| \\
&= |\{(I_1, ..., I_{l-1}, \{k+1\}) : (I_1, ..., I_{l-1}) \in P_{l-1}^k\} \\
&\quad \cup \{(I_1, ..., \tilde{I}_j, ..., I_l) : \tilde{I}_j = I_j \cup \{k+1\}, (I_1, ..., I_l) \in P_l^k\}| \\
&= |P_l^{k+1}|.
\end{aligned}
$$

Thus, we have established $C_{\gamma^k}^{\beta^k} = |P_l^k|$ by the mathematical induction and hence, the conclusion of this lemma.

Proof of Theorem 2. For any $\gamma \in \mathbb{Z}_+^s$ with $|\gamma| \geqslant 2$, we decompose γ into multi-integers $m(i)$ of length 1, $i = 1, ..., |\gamma|$, i.e., $\gamma = m(1) + \cdots + m(|\gamma|)$. Then

$$
\begin{aligned}
\frac{1}{n^{|\gamma|}} S_\gamma^n(\mathbf{x}) &= \sum_{\substack{|\alpha| = n \\ \alpha \in \mathbb{Z}_+^{s+1}}} (\mathbf{x}_\alpha - \mathbf{x})^\gamma B_\alpha(\lambda) \\
&= \sum_{\substack{|\alpha| = n \\ \alpha \in \mathbb{Z}_+^{s+1}}} \left(\frac{1}{n} \sum_{j=0}^{s} \alpha_j (\mathbf{v}^j - \mathbf{x}) \right)^\gamma B_\alpha(\lambda) \\
&= \frac{1}{n^{|\gamma|}} \sum_{\substack{|\alpha| = n \\ \alpha \in \mathbb{Z}_+^{s+1}}} \prod_{i=1}^{|\gamma|} \left(\sum_{j=0}^{s} \alpha_j (\mathbf{v}^j - \mathbf{x})^{m(i)} \right) B_\alpha(\lambda) \\
&= \frac{1}{n^{|\gamma|}} \sum_{\substack{|\alpha| = n \\ \alpha \in \mathbb{Z}_+^{s+1}}} \sum_{i_1, ..., i_{|\gamma|} = 0}^{s} \alpha_{i_1} \cdots \alpha_{i_{|\gamma|}} \\
&\quad \times (\mathbf{v}^{i_1} - \mathbf{x})^{m(1)} \cdots (\mathbf{v}^{i_{|\gamma|}} - \mathbf{x})^{m(|\gamma|)} B_\alpha(\lambda) \\
&= \frac{1}{n^{|\gamma|}} \sum_{i_1, ..., i_{|\gamma|} = 0}^{s} \sum_{\substack{|\alpha| = n \\ \alpha \in \mathbb{Z}_+^{s+1}}} \alpha_{i_1} \cdots \alpha_{i_{|\gamma|}} \\
&\quad \times B_\alpha(\lambda)(\mathbf{v}^{i_1} - \mathbf{x})^{m(1)} \cdots (\mathbf{v}^{i_{|\gamma|}} - \mathbf{x})^{m(|\gamma|)} \\
&= \frac{1}{n^{|\gamma|}} \sum_{i_1, ..., i_{|\gamma|} = 0}^{s} \sum_{\substack{\eta \leqslant \beta(i_1, ..., i_{|\gamma|}); \\ \eta_j \geqslant 1 \text{ if } \beta(i_1, ..., i_{|\gamma|})_j \neq 0}} C_\eta^{\beta(i_1, ..., i_{|\gamma|})} \\
&\quad \times n(n-1) \cdots (n - |\eta| + 1) \\
&\quad \times \lambda^\eta (\mathbf{v}^{i_1} - \mathbf{x})^{m(1)} \cdots (\mathbf{v}^{i_{|\gamma|}} - \mathbf{x})^{m(|\gamma|)}
\end{aligned}
$$

by using Lemma 1. Here $\beta(i_1, ..., i_{|\gamma|}) = \mathbf{e}^{i_1} + \cdots + \mathbf{e}^{i_{|\gamma|}} \in \mathbb{Z}_+^{s+1}$.

By Lemma 2, we may rewrite the second summation of the above expression with respect to the tensor product of partition collections of $\beta(i_1, ..., i_{|\gamma|})$ and interchange the order of summation of the above expression to get

$$\frac{1}{n^{|\gamma|}} S_\gamma^n(\mathbf{x}) = \sum_{k=1}^{|\gamma|} \sum_{\substack{\beta^1,...,\beta^k \in \mathbb{Z}_+^s \\ \beta^1 + \cdots + \beta^k = \gamma}} \frac{n(n-1)\cdots(n-k+1)}{n^{|\gamma|}}$$

$$\times \sum_{j_1,...,j_k=0}^{s} \lambda_{j_1}(\mathbf{v}^{j_1}-\mathbf{x})^{\beta^1} \cdots \lambda_{j_k}(\mathbf{v}^{j_k}-\mathbf{x})^{\beta_k}$$

$$= \sum_{k=1}^{|\gamma|} \sum_{\substack{\beta^1,...,\beta^k \in \mathbb{Z}_+^s \\ \beta^1 + \cdots + \beta^k = \gamma}} \frac{n(n-1)\cdots(n-k+1)}{n^{|\gamma|}}$$

$$\times \prod_{j=1}^{k} \left(\sum_{i=0}^{s} \lambda_i(\mathbf{v}^i - \mathbf{x})^{\beta^j} \right). \qquad (*)$$

Indeed, the summation $\sum_{i_1,...,i_{|\gamma|}=0}^{s}$ can be arranged in such a fashion that for $1 \leqslant k \leqslant \min\{s, |\gamma|\}$, $i_1, ..., i_{|\gamma|}$ are divided in k groups and each of them varies from 0 to s independently and is not equal to the other groups. Then we use Lemma 1 and Lemma 2 and interchange the order of the summation. For example, for $k = 1$,

$$\frac{1}{n^{|\gamma|}} \sum_{i_1 = \cdots = i_{|\gamma|}=0}^{s} \sum_{\substack{\alpha \in \mathbb{Z}_+^{s+1} \\ |\alpha| = n}} \alpha_{i_1} \cdots \alpha_{i_{|\gamma|}}$$

$$\times B_\alpha(\lambda)(\mathbf{v}^{i_1} - \mathbf{x})^{m(1)} \cdots (\mathbf{v}^{i_{|\gamma|}} - \mathbf{x})^{m(|\gamma|)}$$

$$= \frac{1}{n^{|\gamma|}} \sum_{j=0}^{s} \sum_{\substack{|\alpha| = n \\ \alpha \in \mathbb{Z}_+^{s+1}}} \alpha_j^{|\gamma|} B_\alpha(\lambda)(\mathbf{v}^j - \mathbf{x})^\gamma$$

$$= \frac{1}{n^{|\gamma|}} \sum_{j=0}^{s} \sum_{k=1}^{|\gamma|} |P_k^{|\gamma|}| \, n(n-1)\cdots(n-k+1)\, \lambda_j^k(\mathbf{v}^j - \mathbf{x})^\gamma$$

$$= \sum_{k=1}^{|\gamma|} \frac{n(n-1)\cdots(n-k+1)}{n^{|\gamma|}} \sum_{j=0}^{s} |P_k^{|\gamma|}| \, \lambda_j^k(\mathbf{v}^j - x)^\gamma$$

$$= \frac{n}{n^{|\gamma|}} \sum_{j=0}^{s} \lambda_j(\mathbf{v}^j - \mathbf{x})^\gamma + \frac{n(n-1)}{n^{|\gamma|}} \sum_{j=0}^{s} |P_2^{|\gamma|}| \, \lambda_j^2(\mathbf{v}^j - x)^\gamma + \cdots .$$

Note that the first term above is one of the terms in $(*)$. For $k = 2$, consider

$$\frac{1}{n^{|\gamma|}} \sum_{\substack{i_1=0,\, i_2=\cdots=i_{|\gamma|}=0 \\ i_1 \neq i_2}}^{s} \sum_{\substack{\alpha \in \mathbb{Z}_+^{s+1} \\ |\alpha|=n}} \alpha_{i_1} \cdots \alpha_{i_{|\gamma|}}$$

$$\times B_\alpha(\lambda)(\mathbf{v}^{i_1}-\mathbf{x})^{m(1)} \cdots (\mathbf{v}^{i_{|\gamma|}}-\mathbf{x})^{m(|\gamma|)}$$

$$= \frac{1}{n^{|\gamma|}} \sum_{\substack{i=0,\, j=0 \\ i \neq j}}^{s} \sum_{\substack{|\alpha|=n \\ \alpha \in \mathbb{Z}_+^{s+1}}} \alpha_i \alpha_j^{|\gamma|-1}$$

$$\times B_\alpha(\lambda)(\mathbf{v}^i-\mathbf{x})^{m(1)} (\mathbf{v}^j-\mathbf{x})^{\gamma-m(1)}$$

$$= \frac{1}{n^{|\gamma|}} \sum_{\substack{i=0,\, j=0 \\ i \neq j}}^{s} \sum_{(1,k) \leqslant (1,|\gamma|-1)} |P_1^1 \otimes P_k^{|\gamma|-1}|$$

$$\times \lambda_i \lambda_j^k n(n-1) \cdots (n-k)(\mathbf{v}^i-\mathbf{x})^{m(1)} (\mathbf{v}^j-\mathbf{x})^{\gamma-m(1)}$$

$$= \sum_{k=1}^{|\gamma|-1} \frac{n(n-1)\cdots(n-k)}{n^{|\gamma|}} \sum_{\substack{i=0,\, j=0 \\ i \neq j}}^{s} |P_k^{|\gamma|-1}|$$

$$\times \lambda_i \lambda_j^k (\mathbf{v}^i-\mathbf{x})^{m(1)} (\mathbf{v}^j-\mathbf{x})^{\gamma-m(1)}$$

$$= \frac{n(n-1)}{n^{|\gamma|}} \sum_{\substack{i=0,\, j=0 \\ i \neq j}}^{s} \lambda_i \lambda_j (\mathbf{v}^i-\mathbf{x})^{m(1)} (\mathbf{v}^j-\mathbf{x})^{\gamma-m(1)} + \cdots .$$

We can divide $i_1, ..., i_{|\gamma|}$ into the other kind of the two groups and find the corresponding summation. There are $|P_2^{|\gamma|}|$ possibilities of such partitions of indices $i_1, ..., i_{|\gamma|}$. Thus, the second term in the summation for $k=1$ comes into play with the first term in the summation for $k=2$ to become

$$\frac{n(n-1)}{n^{|\gamma|}} \sum_{\beta^1+\beta^2=\gamma} \sum_{i=0,\, j=0}^{s} \lambda_i (\mathbf{v}^i-\mathbf{x})^{\beta^1} \lambda_j (\mathbf{v}^j-\mathbf{x})^{\beta^2},$$

which is one of the terms in the expression $(*)$. These justify some of the terms in $(*)$. Similarly, we can verify the remaining terms. This completes the proof.

Proof of Theorem 3. Since $f \in C^{2k}(T)$, we expand $f(\mathbf{x}_\alpha)$ at $\mathbf{x}$ in Taylor polynomials

$$f(\mathbf{x}_\alpha) = \sum_{j=0}^{2k-1} \frac{1}{j!} D^j_{\mathbf{x}_\alpha-\mathbf{x}} f(\mathbf{x}) + \frac{1}{(2k)!} D^{2k}_{\mathbf{x}_\alpha-\mathbf{x}} f(\psi_\alpha)$$

$$= \sum_{|\gamma| \leqslant 2k} \frac{1}{\gamma!} D^\gamma f(\mathbf{x})(\mathbf{x}_\alpha-\mathbf{x})^\alpha$$

$$+ \frac{1}{(2k)!} (D^{2k}_{\mathbf{x}_\alpha-\mathbf{x}} f(\psi_\alpha) - D^{2k}_{\mathbf{x}_\alpha-\mathbf{x}} f(\mathbf{x})).$$

for some intermediate point ψ_α. Thus,

$$B_n(f;\mathbf{x})-f(\mathbf{x})=\sum_{|\gamma|\leqslant 2k}\frac{1}{\gamma!}D^\gamma f(\mathbf{x})\frac{S_\gamma^n(\mathbf{x})}{n^{|\gamma|}}$$

$$+\sum_{|\gamma|=2k}\frac{1}{\gamma!}\sum_{\substack{|\alpha|=n\\ \alpha\in\mathbb{Z}_+^{s+1}}}(D^\gamma f(\psi_\alpha)-D^\gamma f(\mathbf{x}))(\mathbf{x}_\alpha-\mathbf{x})^\gamma B_\alpha(\lambda).$$

Now we apply Bernstein's argument to the second term of the above expression. Indeed, we consider

$$\sum_{|\mathbf{x}_\alpha-\mathbf{x}|<\delta}+\sum_{|\mathbf{x}_\alpha-\mathbf{x}|\geqslant\delta}(D^\gamma f(\psi_\alpha)-D^\gamma f(\mathbf{x}))(\mathbf{x}_\alpha-\mathbf{x})^\gamma B_\alpha(\lambda)$$

and estimate each of the above two terms separately:

$$\begin{aligned}
\left|\sum_{|\mathbf{x}_\alpha-\mathbf{x}|<\delta}\right| &\leqslant \varepsilon\sum_{\substack{|\alpha|=n\\ |\mathbf{x}_\alpha-\mathbf{x}|<\delta}}|(\mathbf{x}_0-\mathbf{x})^\gamma|\,B_\alpha(\lambda)\\
&\leqslant \varepsilon\sum_{|\alpha|=n}\|\mathbf{x}_\alpha-\mathbf{x}\|_2^{2k}B_\alpha(\lambda)\\
&=\varepsilon\sum_{|\alpha|=n}[(\mathbf{x}_\alpha-\mathbf{x})\cdot(\mathbf{x}_\alpha-\mathbf{x})]^k B_\alpha(\lambda)\\
&=\varepsilon\sum_{|\alpha|=n}\frac{1}{n^{2k}}\left(\sum_{i=1}^{s}\sum_{l,m=0}^{s}\alpha_l\alpha_m(\mathbf{v}^l-\mathbf{x})_i(\mathbf{v}^m-\mathbf{x})_i\right)^k B_\alpha(\lambda)\\
&=\frac{\varepsilon}{n^{2k}}\sum_{j_1,\ldots,j_k=1}^{s}\sum_{i_1,\ldots,i_{2k}=0}^{s}\sum_{|\alpha|=n}\alpha_{i_1}\cdots\alpha_{i_{2k}}B_\alpha(\lambda)\\
&\quad\times(\mathbf{v}^{i_1}-\mathbf{x})^{m(j_1)}(\mathbf{v}^{i_2}-\mathbf{x})^{m(j_1)}\cdots(\mathbf{v}^{i_{2k-1}}-\mathbf{x})^{m(j_k)}(\mathbf{v}^{i_{2k}}-\mathbf{x})^{m(j_k)}\\
&=\varepsilon\sum_{j_1,\ldots,j_k=1}^{s}\sum_{i_1,\ldots,i_{2k}=0}^{s}\sum_{\substack{\eta\leqslant\beta(i_1,\ldots,i_{2k})\\ \eta_l\geqslant 1 \text{ if } \beta(i_1,\ldots,i_{2k})_l\neq 0}}C_\eta^{\beta(i_1,\ldots,i_{2k})}\\
&\quad\times\frac{n(n-1)\cdots(n-|\eta|+1)}{n^{2k}}\lambda^\eta(\mathbf{v}^{i_1}-\mathbf{x})^{m(j_1)}\\
&\quad\times(\mathbf{v}^{i_2}-\mathbf{x})^{m(j_1)}\cdots(\mathbf{v}^{i_{2k-1}}-\mathbf{x})^{m(j_k)}(\mathbf{v}^{i_{2k}}-\mathbf{x})^{m(j_k)}.
\end{aligned}$$

Here, $m(i)=\mathbf{e}^i\in\mathbb{Z}_+^s$ is a standard unit vector with 1 in the ith component and 0 otherwise, $i=1,\ldots,s$.

Let $\gamma(j_1,\ldots,j_k)=2m(j_1)+\cdots+2m(j_k)\in\mathbb{Z}_+^s$. As in the proof of Theorem 2, we rewrite the third summation in terms of the tensor product of partition collections of $\beta(i_1, ..., i_{2k})$ and interchange the order of the second and third summations of the above expression to get

$$\left|\sum_{|\mathbf{x}_\alpha - \mathbf{x}| < \delta}\right| \leqslant \varepsilon \sum_{j_1, ..., j_k = 1}^{s} \sum_{\substack{\beta^1, ..., \beta^i \in \mathbb{Z}_+^s \\ \beta^1 + \cdots + \beta^i = \gamma(j_1, ..., j_k)}} \frac{n(n-1) \cdots (n-i+1)}{n^{2k}}$$

$$\times \sum_{l_1, ..., l_i = 0}^{s} \lambda_{l_1}(\mathbf{v}^{l_1} - \mathbf{x})^{\beta^1} \cdots \lambda_{l_i}(\mathbf{v}^{l_i} - \mathbf{x})^{\beta^i}$$

$$= \varepsilon \sum_{j_1, ..., j_k = 1}^{s} \sum_{\substack{\beta^1, ..., \beta^i \in \mathbb{Z}_+^s \\ \beta^1 + \cdots + \beta^i = \gamma(j_1, ..., j_k) \\ |\beta^l| \geqslant 2,\ l = 1, ..., i}} \frac{n(n-1) \cdots (n-i+1)}{n^{2k}}$$

$$\times \prod_{l=1}^{s} \left(\sum_{j=0}^{s} \lambda_j (\mathbf{v}^j - \mathbf{x})^{\beta^l} \right).$$

Here we have used the fact that $\sum_{i=0}^{s} \lambda_i (\mathbf{v}^i - \mathbf{x})^{m(j)} = 0$ for any $j = 1, ..., s$. Hence,

$$\left|\sum_{|\mathbf{x}_\alpha - \mathbf{x}| < \delta}\right| \leqslant \varepsilon O\left(\frac{1}{n^k}\right).$$

Furthermore, letting $M_f := 2 \max\{\|D^{2k} f\|_{L_\infty(T)}\}$, we have

$$\left|\sum_{|\mathbf{x}_\alpha - \mathbf{x}| \geqslant \delta}\right| \leqslant \frac{M_f}{\delta^2} \sum_{|\alpha| = n} |(\mathbf{x}_\alpha - \mathbf{x})^\gamma| \; \|\mathbf{x}_\alpha - \mathbf{x}\|_2^2 B_\alpha(\lambda)$$

$$\leqslant \frac{M_f}{\delta^2} \sum_{|\alpha| = n} [(\mathbf{x}_\alpha - \mathbf{x}) \cdot (\mathbf{x}_\alpha - \mathbf{x})]^{k+1} B_\alpha(\lambda).$$

As before, we can prove

$$\sum_{|\alpha| = n} [(\mathbf{x}_\alpha - \mathbf{x}) \cdot (\mathbf{x}_\alpha - \mathbf{x})]^{k+1} B_\alpha(\lambda) = O\left(\frac{1}{n^{k+1}}\right).$$

With these two estimates and an application of the corollary of Theorem 2, we obtain the result of Theorem 3.

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